UNIVERSIDADE FEDERAL SANTA MARIA
DEPARTAMENTO DE FÍSICA
Programa de Pós-Graduação em Física
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FSC 810 Mecânica Estatística

List 6

1) For a system of non-interacting indistinguishable particles obeying Bose-Einstein, Fermi-Dirac or Maxwell-Boltzmann statistics, obtain the probability $P_i(n_i, T, \mu)$ for finding n_i particles in a given single-particle state, labelled by *i,* when the system is in equilibrium with a particle reservoir with temperature T and chemical potential μ. Making use of this distribution, find the average occupation number $\langle n_i \rangle$. Compute the relative fluctuation in the occupation number $\frac{\Delta n_i}{\langle n_i \rangle}$, where $\Delta n_i = \sqrt{\langle (n_i - \langle n_i \rangle) \rangle^2}$.

2) Show that the entropy of an ideal quantum gas may be written as

$$S = -k_B \sum_j \{ f_j \ln f_j \pm (1 \mp f_j) \ln (1 \mp f_j) \}$$

where the upper (lower) sign refers to fermions (bosons), and

$$f_j = \langle n_j \rangle = \frac{1}{\exp[\beta(\epsilon_j - \mu)] \pm 1}$$

is the Fermi-Dirac (Bose-Einstein) distribution.

3) Show that the equation of state pV=(2/3)U holds for both free bosons and fermions. Show that an ideal ultrarelativistic gas, given by energy spectrum $\epsilon = c \hbar k$, still obeys the same equation of state.

4) An electron in a magnetic field H has energy $\varepsilon \mp \mu_B H$, where ε is its kinetic energy and μ_B is the Bohr magneton, depending on whether its magnetic moment is parallel or antiparallel to the field. Find the zero field paramagnetic susceptibility of a gas of electrons, including the leading correction to the zero-temperature value.

5) Show that the chemical potential of an ideal classical gas of N monoatomic particles, in a container of volume V, at temperature T, may be written as

$$\mu = k_B T \ln\left(\frac{\lambda^3}{v}\right)$$

where v=V/N is the volume per particle, and $\lambda = h/\sqrt{2\pi m k_B T}$
is the thermal wavelength. Obtain the first
quantum correction to this result. That is, show that the chemical potential of a ideal quantum gas

may be written as the expansion

$$\frac{\mu}{k_B T} - \ln\left(\frac{\lambda^3}{v}\right) = A \ln\left(\frac{\lambda^3}{v}\right) + B \ln\left(\frac{\lambda^3}{v}\right)^2 + \ldots$$

and obtain
explicit expressions for the prefactor A for fermions and bosons. Sketch a graph of

$\mu/k_B T$ versus λ^2 for fermions, bosons, and classical particles.

6. Consider a gas of N free electrons, in a region of volume V, in the ultrarelativistic regime. The energy spectrum is given by

$$\epsilon = \left[p^2c^2 + m^2c^4\right]^{1/2} \approx pc,$$

where $\vec{p}$ is the linear momentum.

(a) Calculate the Fermi energy of this system.

(b) What is the total energy in the ground state?

(c) Obtain an asymptotic form for the specific heat at constant volume in the limit $T << T_F$.

7. At low temperatures, the internal energy of a system of free electrons may be written as an expansion,

$$U = \frac{3}{5}N\epsilon_F\left\{1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 - A\left(\frac{T}{T_F}\right)^4 + \ldots\right\}.$$

Obtain the value of the constant A, and indicate the order of magnitude of the terms that have been discarded.

8. Consider a system of free fermions in d dimensions, with the energy spectrum

$$\epsilon_{\vec{k},\sigma} = c\left|\vec{k}\right|^a,$$

where $c > 0$ and $a > 1$.

(a) Calculate the prefactor A of the relation $pV = AU$.

(b) Calculate the Fermi energy as a function of volume V and number of particles N.

(c) Calculate an asymptotic expression, in the limit $T << T_F$, for the specific heat at constant volume.

9) Consider an ideal gas of bosons with internal degrees of freedom. Suppose that, besides the ground state with zero energy ($\epsilon_o = 0$), we have to take into account the first excited state, with internal energy $\epsilon_1 > 0$. In other words, assume that the energy spectrum is given by

$$\epsilon_j = \epsilon_{\vec{k},\sigma} = \frac{\hbar^2 k^2}{2m} + \epsilon_1 \sigma,$$

where $\sigma = 0, 1$. Obtain an expression for the Bose–Einstein condensation temperature as a function of the parameter ϵ_1.

10) Consider a gas of non-interacting bosons associated with the energy spectrum

$$\epsilon = \hbar c\left|\vec{k}\right|,$$

where $\hbar$ and c are constants, and $\vec{k}$ is a wave vector. Calculate the pressure of this gas at zero chemical potential. What is the pressure of radiation of a gas of photons?